Numerical dosimetry of ELF fields by using dual formulations

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Introduction

The numerical computation of electromagnetic fields induced into the human body by environmental ELF fields depends not only on the method used but also on the formulation. When dealing with, e.g. the Finite Element (FE) method, the choice of the formulation has an influence on the results. Even though the solution of any formulation tends anymptoticaly to the (unique) solution of the electromagnetic problem, a suitable choice of formulations allows to estimate a bound on electrical quantities. The aim of this paper is thus to determine this bound by means of two dual formulations.

Materials and Methods

In the context of ELF magnetic fields, Maxwell's equations can be simplified as:

$$\operatorname{curl}\mathbf{e} = \partial_t \mathbf{b}\,, \quad \operatorname{curl}\mathbf{h} = \mathbf{j}\,, \quad \operatorname{div}\mathbf{b} = 0\,, \tag{1 a, b, c}$$

with $\mathbf{e}$ the electric field, $\mathbf{h}$ the magnetic field, $\mathbf{j}$ the electric current density and $\mathbf{b}$ the magnetic flux density. The problem is fully determined by adding the constitutive laws: $\mathbf{j} = \sigma\mathbf{e}$, $\mathbf{b} = \mu\mathbf{h}$ (with conductivity σ, permeability μ) and the appropriate boundary condition: $\mathbf{n}\cdot\mathbf{j}|_{\partial\Omega} = 0$.

The computational domain Ω is limited to the human body, the influence of which on the environmental magnetic field is supposed to be negligible [1]. At the continuous level, these fields are welcome into a structure formed by two dual de Rahm's complexes joined by the constitutive laws, as represented in the followng Tonti's diagram:

$$\begin{array}{ccccccc} \phi & \xrightarrow{\text{grad}} & \mathbf{e},\mathbf{a} & \xrightarrow{\text{curl}} & \mathbf{b} & \xrightarrow{\text{div}} & 0 \\ & & \sigma\updownarrow & & \mu\updownarrow & & \\ 0 & \xleftarrow{\text{div}} & \mathbf{j} & \xleftarrow{\text{curl}} & \mathbf{h},\mathbf{t} & & \end{array} \tag{2}$$

Unfortunately at the discrete level it is impossible to fulfill exactly all the equations. By using standard (i.e. not mixed) formulations the constitutive laws and the equation on one of the two levels of Tonti's diagram are enforced strongly, whereas the equations on the other level cannot be imposed but in a weak sense [2].

The e–conform $\phi - \mathbf{a}$ formulation

One possibility is to enforce the upper level of (2), i.e. to impose strongly Faraday's law (1a). Let $\mathbf{a}\,:\,\operatorname{curl}\mathbf{a} = \mathbf{b}$ be a magnetic vector potential, which is supposed to be known *a priori*. By using (1a) one obtains that: $\mathbf{e} = -\partial_t\mathbf{a} - \operatorname{grad}\phi$, where ϕ is an unknown electric scalar potential. The weak form of Ampère's law (1b) reads:

$$(\sigma(\partial_t\mathbf{a} + \operatorname{grad}\phi),\ \operatorname{grad}\phi') = 0 \quad \forall\phi' \in \mathrm{H}(\text{grad},\Omega) \tag{3}$$

The j–conform $\mathbf{t} - \mathbf{b}$ formulation

Analogously, we can strongly enforce the lower level of (2), i.e. the divergence of Ampère's law (1b), $\operatorname{div}\mathbf{j} = 0$. Let $\mathbf{t}\,:\,\operatorname{curl}\mathbf{t} = \mathbf{j}$ be an unknown electric vector potential. The weak form of Faraday's law (1a) is given by

$$(\frac{1}{\sigma}\operatorname{curl}\mathbf{t},\ \operatorname{curl}\mathbf{t}') + (\partial_t\mathbf{b},\ \mathbf{t}') = 0 \quad \forall\mathbf{t}' \in \mathbf{H}_0(\text{curl},\Omega) \tag{4}$$

Bilateral bounding of the co-energy

These two formulations disregard the induction reaction but still allow to compute the eddy currents: indeed they have the mathematical structure of a static problem. Existing results on error bounding can thus be straightforwardly extended to the context of ELF dosimetry.

Let $\mathbf{e}^*$ be the (discrete) electric field computed by (3), and $\mathbf{j}^*$ be the (discrete) current density computed by (4). The co-energy is defined in terms of $\mathbf{e}$ ($\mathscr{E}_e^C$) and $\mathbf{j}$ ($\mathscr{E}_j^C$), respectively, as:

$$\mathscr{E}_e^C = \int_\Omega \int_0^{\mathbf{e}^*} \mathbf{j}\, d\mathbf{e}\,, \qquad \mathscr{E}_j^C = \int_\Omega \mathbf{e}^* \cdot \mathbf{j}^* - \int_\Omega \int_0^{\mathbf{j}^*} \mathbf{e}\, d\mathbf{j}\,. \tag{5}$$

Assuming σ constant, $\mathscr{E}_e^C$ can be computed as a function of $\mathbf{e}^*$: $\mathscr{E}_e^C = \frac{1}{2}(\sigma\mathbf{e}^*,\, \mathbf{e}^*)$. Analogously $\mathscr{E}_j^C$ can be computed as a function of $\mathbf{j}^*$: $\mathscr{E}_j^C = (-\partial_t \mathbf{a},\, \mathbf{j}^*) - \frac{1}{2}(\frac{1}{\sigma}\mathbf{j}^*,\, \mathbf{j}^*)$. At the discrete level, the constitutive law $\mathbf{j}^* \neq \sigma\mathbf{e}^*$ is not exactly satisfied, so that: $\mathscr{E}_e^C - \mathscr{E}_j^C \geq 0$. In particular $\mathscr{E}_e^C = \mathscr{E}_j^C$ holds only if $\mathbf{j} = \sigma\mathbf{e}$, the bound for the *exact* coenergy $\mathscr{E}^C$ is thus given by:

$$\mathscr{E}_e^C \geq \mathscr{E}^C \geq \mathscr{E}_j^C\,. \tag{6}$$

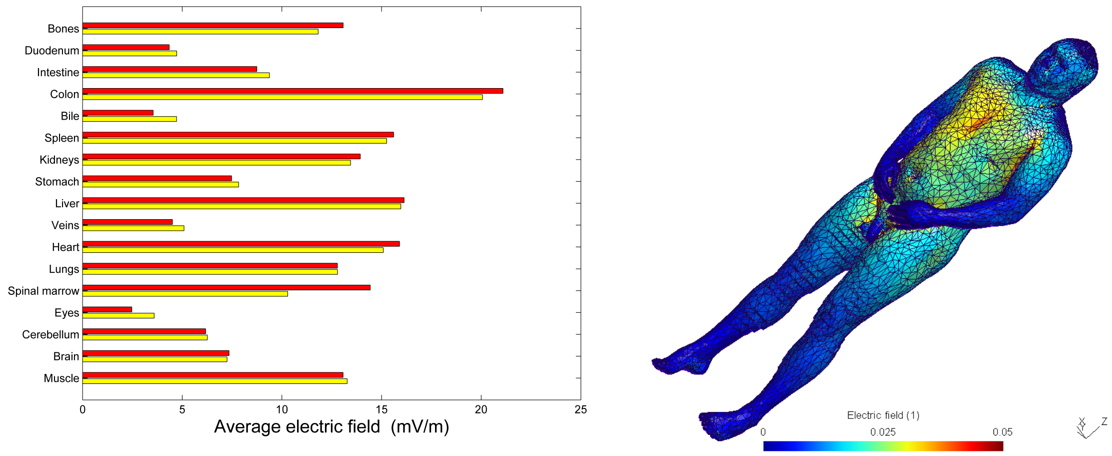

Figure 1: Left: average electric field computed with the $\phi - \mathbf{a}$ (red) and the $\mathbf{t} - \mathbf{b}$ (yellow) formulations. Right: electric field at the surface of the phantom.

Results

Both formulations (3) and (4) have been discretized with Whitney elements by means of the software GetDP [3]. We simulated the exposure to a uniform vertical field $B_z = 1\,\mathrm{mT}$ at 50 Hz. The computational phantom is based on the Visible Human project. Note that the bound (6) holds only for the global co-energy $\mathscr{E}^C$, and not for the local fields $\mathbf{e}$ or $\mathbf{j}$. However, the average values of the electric field computed on the different organs (figure 1) suggest that in most cases the $\phi - \mathbf{a}$ formulation overestimates values, conversely to the $\mathbf{t} - \mathbf{b}$ formulation.

Summary and Conclusions

The use of dual formulations allows to numerically bound the electrical co-energy. Moreover, the error in the constitutive law may be exploited for adaptive strategies.

References

[1] R. Scorretti, R. Perrussel, L. Morel, N. Burais, L. Nicolas Numerical dosimetry of currents induced in the human body by ELF magnetic fields. Accepted in *COMPEL*.

[2] A. Bossavit A rationale for edge-elements in 3-D fields computations. *IEEE Trans. Mag.*, vol. 24, no. 1, 74–79, 1988.

[3] P. Dular, C. Geuzaine, F. Henrotte, W. Legros. A general environment for the treatment of discrete problems and its application to the finite element method. *IEEE Trans. Mag.*, vol. 34, no. 5, 3395–3398, 2002.